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Higher-order parameter-free sufficient optimality conditions in discrete minmax fractional programming

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Abstract

The purpose of this paper is to establish a fairly large number of sets of second-order parameter-free sufficient optimality conditions for a discrete minmax fractional programming problem. Our effort to accomplish this goal is by utilizing various new classes of generalized second-order $(\varphi, \eta, \rho, \theta, m)$-invex functions, which generalize most of the concepts available in the literature.

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1 Introduction

Recently, Verma and Zalmai [26] investigated the second-order parametric necessary optimality constraints as well as sufficient optimality constraints for a discrete minmax fractional programming problem applying generalized second order invex functions. Mathematical fractional programming problems of this nature with a finite number of variables as well as finite number of constraints are referred to as generalized fractional programming problems, while mathematical fractional programming problems with a finite number of variables and infinitely many constraints are called *semiinfinte fractional programming problems* in the literature. These problems, among real-world applications, have a wide range of significant applications, for example, consider a robot problem. Then a class of control problems in robotic can be handled by a semi-infinite program, especially the maneuverability problem (Lopez and Still [9]) for the generalized semi-infinite program. It is highly probable that among all industries, especially for the automobile industry, the robots are about to revolutionize the assembly plants forever. That would change the face of other industries toward technical innovation as well.

On the other hand, just recently Zalmai and Zhang [39] presented a simpler proof to a theorem of the alternative (Luu and Hung [10]) by using a separation theorem and used Dini and Hadamard directional derivatives and differentials under Karush-Kuhn-Tucker-type necessary efficiency conditions for a semiinfinite multiobjective optimization problem on a normed linear space. There exists a large volume of research publications on mathematical programming/ mathematical fractional programming based on using the first derivatives, Dini derivatives, Hadamard derivatives, F-derivatives, G-derivatives, and second derivatives in the literature, but it would be an opportune juncture to explore to fractional derivatives to the context of general mathematical programming problems. For more details on fractional derivatives and their applications, we refer the reader to Srivastava and Associates [7] - [8]. It is quite remarkable that there are more than one million research publications on fractional differential equations and related applications to other fields alone

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in the literature. Motivated (and greatly impacted) by these research advances and existing scopes for interdisciplinary research, we consider the following discrete minmax fractional programming problem:

$$(P) \qquad \text{Minimize} \max_{1\leq i\leq p} \frac{f_i(x)}{g_i(x)}$$

subject to $\quad G_j(x) \leq 0,\ j \in \underline{q}, \quad H_k(x) = 0,\ k \in \underline{r}, \quad x \in X,$

where X is an open convex subset of $\mathbb{R}^n$ (n-dimensional Euclidean space), $f_i, g_i, i \in \underline{p} \equiv \{1, 2, \ldots, p\}$, $G_j, j \in \underline{q}$, and $H_k, k \in \underline{r}$, are real-valued functions defined on X, and for each $i \in \underline{p}$, $g_i(x) > 0$ for all x satisfying the constraints of (P).

In this paper, we plan to investigate based on the fast developing field of the mathematical programming, some parameter-free new classes of second-order parameter-free optimality results using the second-order invex functions. The results established in this paper can further be utilized for formulating and proving numerous second-order parameter-free duality theorems for the discrete minmax fractional programming problem (P). The obtained results are new and encompass most of the results on generalized invex functions in the literature (including [1]-[6], [11]-[33], [35]-[40]). Furthermore, our results can (based on Pitea and Postolache [17]) be applied to a new class of multitime multiobjective variational problems for minimizing a vector of functional of curvilinear type to the context of the generalized Mond-Weir-Zalmai type quasiinvexity. For more details, we refer the reader [1]-[40].

The rest of this paper is organized as follows. In the remainder of this section, we recall a few basic definitions and auxiliary results which will be needed in the sequel. In Section 2, we state and prove a multiplicity of second-order parameter-free sufficient optimality results for (P) using a variety of generalized $(\varphi, \eta, \rho, \theta, m)$-sonvexity assumptions. Finally, in Section 3, we summarize our main results and also point out some future research endeavors arising from certain modifications of the principal problem investigated in the present paper.

Note that all the optimality results established for (P) are also applicable, when appropriately specialized, to the following three classes of problems with discrete max, fractional, and conventional objective functions, which are particular cases of (P):

$$(P1) \qquad \underset{x\in\mathbb{F}}{\text{Minimize}} \max_{1\leq i\leq p} f_i(x);$$

$$(P2) \qquad \underset{x\in\mathbb{F}}{\text{Minimize}} \frac{f_1(x)}{g_1(x)};$$

$$(P3) \qquad \underset{x\in\mathbb{F}}{\text{Minimize}}\ f_1(x),$$

where $\mathbb{F}$ is the feasible set of (P), that is,

$$\mathbb{F} = \{x \in X : G_j(x) \leq 0,\ j \in \underline{q}, \quad H_k(x) = 0, \quad k \in \underline{r}\}.$$

We next recall some basic concepts for certain classes of generalized convex functions, introduced recently in [26], which will be needed in the sequel. We shall refer the 'second-order invex functions' to as the sonvex functions. Let $f : X \to \mathbb{R}$ be a twice differentiable function.

Definition 1.1. The function f is said to be *(strictly)* $(\varphi, \eta, \rho, \theta, m)$*-sonvex* at x^* if there exist functions $\varphi : \mathbb{R} \to \mathbb{R}$, $\eta : X \times X \to \mathbb{R}^n$, $\rho : X \times X \to \mathbb{R}$, and $\theta : X \times X \to \mathbb{R}^n$, and a positive integer m such that for each $x \in X$ $(x \neq x^*)$ and $z \in \mathbb{R}^n$,

$$\varphi\big(f(x) - f(x^*)\big)(>) \geq \langle \nabla f(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*) z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^m,$$

where $\|\cdot\|$ is a norm on $\mathbb{R}^n$.

The function f is said to be *(strictly)* $(\varphi, \eta, \rho, \theta, m)$*-sonvex* on X if it is *(strictly)* $(\varphi, \eta, \rho, \theta, m)$*-sonvex* at each $x^* \in X$.

Definition 1.2. The function f is said to be *(strictly)* $(\varphi, \eta, \rho, \theta, m)$*-pseudosonvex* at x^* if there exist functions $\varphi : \mathbb{R} \to \mathbb{R}$, $\eta : X \times X \to \mathbb{R}^n$, $\rho : X \times X \to \mathbb{R}$, and $\theta : X \times X \to \mathbb{R}^n$, and a positive integer m such that for each $x \in X$ $(x \neq x^*)$ and $z \in \mathbb{R}^n$,

$$\langle \nabla f(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*) z\rangle \geq -\rho(x, x^*)\|\theta(x, x^*)\|^m \;\;\Rightarrow\;\; \varphi\big(f(x) - f(x^*)\big)(>) \geq 0,$$

equivalently,

$$\varphi\big(f(x) - f(x^*)\big)(\leq) < 0 \Rightarrow \langle \nabla f(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*) z\rangle < -\rho(x, x^*)\|\theta(x, x^*)\|^m.$$

The function f is said to be *(strictly)* $(\varphi, \eta, \rho, \theta, m)$*-pseudosonvex* on X if it is *(strictly)* $(\varphi, \eta, \rho, \theta, m)$*-pseudosonvex* at each $x^* \in X$.

Definition 1.3. The function f is said to be *prestrictly* $(\varphi, \eta, \rho, \theta, m)$*-pseudosonvex* at x^* if there exist functions $\varphi : \mathbb{R} \to \mathbb{R}$, $\eta : X \times X \to \mathbb{R}^n$, $\rho : X \times X \to \mathbb{R}$, and $\theta : X \times X \to \mathbb{R}^n$, and a positive integer m such that for each $x \in X$ $(x \neq x^*)$ and $z \in \mathbb{R}^n$,

$$\langle \nabla f(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*) z\rangle > -\rho(x, x^*)\|\theta(x, x^*)\|^m \;\;\Rightarrow\;\; \varphi\big(f(x) - f(x^*)\big) \geq 0,$$

equivalently,

$$\varphi\big(f(x) - f(x^*)\big) < 0 \Rightarrow \langle \nabla f(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*) z\rangle \leq -\rho(x, x^*)\|\theta(x, x^*)\|^m.$$

The function f is said to be *prestrictly* $(\varphi, \eta, \rho, \theta, m)$*-pseudosonvex* on X if it is *prestrictly* $(\varphi, \eta, \rho, \theta, m)$*-pseudosonvex* at each $x^* \in X$.

Definition 1.4. The function f is said to be *(prestrictly)* $(\varphi, \eta, \rho, \theta, m)$*-quasisonvex* at x^* if there exist functions $\varphi : \mathbb{R} \to \mathbb{R}$, $\eta : X \times X \to \mathbb{R}^n$, $\rho : X \times X \to \mathbb{R}$, and $\theta : X \times X \to \mathbb{R}^n$, and a positive integer m such that for each $x \in X$ and $z \in \mathbb{R}^n$,

$$\varphi\big(f(x) - f(x^*)\big)(<) \leq 0 \;\;\Rightarrow\;\; \langle \nabla f(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*) z\rangle \leq -\rho(x, x^*)\|\theta(x, x^*)\|^m,$$

equivalently,

$$\langle \nabla f(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*) z\rangle > -\rho(x, x^*)\|\theta(x, x^*)\|^m \Rightarrow \varphi\big(f(x) - f(x^*)\big)(\geq) > 0.$$

The function f is said to be *(prestrictly)* $(\varphi, \eta, \rho, \theta, m)$*-quasisonvex* on X if it is *(prestrictly)* $(\varphi, \eta, \rho, \theta, m)$*-quasisonvex* at each $x^* \in X$.

Definition 1.5. The function f is said to be *strictly* $(\varphi,\eta,\rho,\theta,m)$*-quasisonvex* at x^* if there exist functions $\varphi:\mathbb{R}\to\mathbb{R}$, $\eta:X\times X\to\mathbb{R}^n$, $\rho:X\times X\to\mathbb{R}$, and $\theta:X\times X\to\mathbb{R}^n$, and a positive integer m such that for each $x\in X$ and $z\in\mathbb{R}^n$,

$$\varphi\big(f(x)-f(x^*)\big)\leq 0 \;\;\Rightarrow\;\; \langle\nabla f(x^*),\eta(x,x^*)\rangle+\frac{1}{2}\langle z,\nabla^2 f(x^*)z\rangle<-\rho(x,x^*)\|\theta(x,x^*)\|^m,$$

equivalently,

$$\langle\nabla f(x^*),\eta(x,x^*)\rangle+\frac{1}{2}\langle z,\nabla^2 f(x^*)z\rangle\geq-\rho(x,x^*)\|\theta(x,x^*)\|^m\Rightarrow\varphi\big(f(x)-f(x^*)\big)>0.$$

The function f is said to be *strictly* $(\varphi,\eta,\rho,\theta,m)$*-quasisonvex* on X if it is *strictly* $(\varphi,\eta,\rho,\theta,m)$*-quasisonvex* at each $x^*\in X$.

From the above definitions it is clear that if f is $(\varphi,\eta,\rho,\theta,m)$-sonvex at x^*, then it is both $(\varphi,\eta,\rho,\theta,m)$-pseudosonvex and $(\varphi,\eta,\rho,\theta,m)$-quasisonvex at x^*, if f is $(\varphi,\eta,\rho,\theta,m)$-quasisonvex at x^*, then it is prestrictly $(\varphi,\eta,\rho,\theta,m)$-quasisonvex at x^*, and if f is strictly $(\varphi,\eta,\rho,\theta,m)$-pseudosonvex at x^*, then it is $(\varphi,\eta,\rho,\theta,m)$-quasisonvex at x^*.

In the proofs of the duality theorems, sometimes it may be more convenient to use certain alternative but equivalent forms of the above definitions. These are obtained by considering the contrapositive statements. Note that the new classes of generalized convex functions specified in Definitions 1.1 - 1.3 contain a variety of special cases that can easily be identified by appropriate choices of φ, η, ρ, θ, and m.

We conclude this section by recalling a set of second-order parameter-free necessary optimality conditions for (P) from the publication [26]. This following result is obtained from Theorem 3.1 of [26] by eliminating the parameter λ^* and redefining the Lagrange multipliers.

Theorem 1.6. [26] Let x^* be an optimal solution of (P) and assume that the functions f_i, g_i, $i\in\underline{p}$, G_j, $j\in\underline{q}$, and H_k, $k\in\underline{r}$, are twice continuously differentiable at x^*, and that the second-order Guignard constraint qualification holds at x^*. Then for each $z^*\in C(x^*)$, there exist $u^*\in U\equiv\{u\in\mathbb{R}^p: u\geq 0,\ \sum_{i=1}^p u_i=1\}$, $v^*\in\mathbb{R}^q_+\equiv\{v\in\mathbb{R}^q: v\geq 0\}$, and $w^*\in\mathbb{R}^r$ such that

$$\sum_{i=1}^p u_i^*[D(x^*,u^*)\nabla f_i(x^*)-N(x^*,u^*)\nabla g_i(x^*)]+\sum_{j=1}^q v_j^*\nabla G_j(x^*)+\sum_{k=1}^r w_k^*\nabla H_k(x^*)=0,$$

$$\Big\langle z^*,\Big\{\sum_{i=1}^p u_i^*[D(x^*,u^*)\nabla^2 f_i(x^*)-N(x^*,u^*)\nabla^2 g_i(x^*)] \\ +\sum_{j=1}^q v_j^*\nabla^2 G_j(x^*)+\sum_{k=1}^r w_k^*\nabla^2 H_k(x^*)\Big\}z^*\Big\rangle\geq 0,$$

$$u_i^*[D(x^*,u^*)f_i(x^*)-N(x^*,u^*)g_i(x^*)]=0,\;\; i\in\underline{p},$$

$$\max_{1\leq i\leq p}\frac{f_i(x^*)}{g_i(x^*)}=\frac{N(x^*,u^*)}{D(x^*,u^*)},$$

$$v_j^* G_j(x^*) = 0, \quad j \in \underline{q},$$

where $C(x^*)$ is the set of all critical directions of (P) at x^*, that is,

$$C(x^*) = \{z \in \mathbb{R}^n : \langle \nabla f_i(x^*) - \lambda \nabla g_i(x^*), z \rangle = 0,\ i \in A(x^*),\ \langle \nabla G_j(x^*), z \rangle \leq 0,\ j \in B(x^*), \\ \langle \nabla H_k(x^*), z \rangle = 0,\ k \in \underline{r}\},$$

$A(x^*) = \{j \in \underline{p} : f_j(x^*)/g_j(x^*) = \max_{1 \leq i \leq p} f_i(x^*)/g_i(x^*)\}$,
$B(x^*) = \{j \in \underline{q} : G_j(x^*) = 0\}$,
$N(x^*, u^*) = \sum_{i=1}^p u_i^* f_i(x^*)$,
and
$D(x^*, u^*) = \sum_{i=1}^p u_i^* g_i(x^*)$.

For brevity, we shall henceforth refer to x^* as a *normal* optimal solution of (P) if it is an optimal solution and satisfies the second-order Guignard constraint qualification.

The form and features of this optimality result will provide clear guidelines for formulating numerous second-order parameter-free sufficient optimality conditions for (P).

2 Sufficient optimality conditions

In this section, we present a fairly large number of second-order parameter-free sufficiency results in which various generalized $(\varphi, \eta, \rho, \theta, m)$-sonvexity assumptions are imposed on the individual as well as certain combinations of the problem functions.

For the sake of the compactness of expressions, we shall use the following list of symbols during the statements and proofs of our sufficiency theorems:

$$D(x, u) = \sum_{i=1}^p u_i g_i(x),$$

$$N(x, u) = \sum_{i=1}^p u_i f_i(x),$$

$$\mathcal{C}(x, v) = \sum_{j=1}^q v_j G_j(x),$$

$$\mathcal{D}_k(x, w) = w_k H_k(x), \quad k \in \underline{r},$$

$$\mathcal{D}(x, w) = \sum_{k=1}^r w_k H_k(x),$$

$$\mathcal{E}_i(x, y, u, \lambda) = D(y, u) f_i(x) - N(y, u) g_i(x), \quad i \in \underline{p},$$

$$\mathcal{E}(x, y, u, \lambda) = \sum_{i=1}^p u_i [D(y, u) f_i(x) - N(y, u) g_i(x)],$$

$$\mathcal{G}(x, v, w) = \sum_{j=1}^q v_j G_j(x) + \sum_{k=1}^r w_k H_k(x),$$

$$I_+(u) = \{i \in \underline{p} : u_i > 0\}, \quad J_+(v) = \{j \in \underline{q} : v_j > 0\}, \quad K_*(w) = \{k \in \underline{r} : w_k \neq 0\}.$$

In the proofs of our sufficiency theorems, we shall make frequent use of the following auxiliary result which provides an alternative expression for the objective function of (P).

Lemma 2.1. [26] For each $x \in X$,

$$\varphi(x) \equiv \max_{1 \leq i \leq p} \frac{f_i(x)}{g_i(x)} = \max_{u \in U} \frac{\sum_{i=1}^{p} u_i f_i(x)}{\sum_{i=1}^{p} u_i g_i(x)}.$$

Theorem 2.2. Let $x^* \in \mathbb{F}$ and assume that the functions f_i, g_i, $i \in \underline{p}$, G_j, $j \in \underline{q}$, and H_k, $k \in \underline{r}$, are twice differentiable at x^*, and that for each $z^* \in C(x^*)$, there exist $u^* \in U$, $v^* \in \mathbb{R}^q_+$, and $w^* \in \mathbb{R}^r$ such that $D(x^*, u^*) > 0$, $N(x^*, u^*) \geq 0$, and

$$\sum_{i=1}^{p} u_i^* [D(x^*, u^*) \nabla f_i(x^*) - N(x^*, u^*) \nabla g_i(x^*)] + \sum_{j=1}^{q} v_j^* \nabla G_j(x^*) + \sum_{k=1}^{r} w_k^* \nabla H_k(x^*) = 0, \tag{2.1}$$

$$\Big\langle z^*, \Big\{ \sum_{i=1}^{p} u_i^* [D(x^*, u^*) \nabla^2 f_i(x^*) - N(x^*, u^*) \nabla^2 g_i(x^*)] + \sum_{j=1}^{q} v_j^* \nabla^2 G_j(x^*) + \sum_{k=1}^{r} w_k^* \nabla^2 H_k(x^*) \Big\} z^* \Big\rangle \geq 0, \tag{2.2}$$

$$u_i^* [D(x^*, u^*) f_i(x^*) - N(x^*, u^*) g_i(x^*)] = 0, \quad i \in \underline{p}, \tag{2.3}$$

$$\max_{1 \leq i \leq p} \frac{f_i(x^*)}{g_i(x^*)} = \frac{N(x^*, u^*)}{D(x^*, u^*)}, \tag{2.4}$$

$$v_j^* G_j(x^*) = 0, \quad j \in \underline{q},$$

$$w_k^* H_k(x^*) \geq 0, \quad k \in \underline{r}. \tag{2.5}$$

Assume, furthermore, that any one of the following six sets of conditions holds:

(a) (i) for each $i \in I_+ \equiv I_+(u)$, f_i is $(\varphi, \eta, \bar{\rho}_i, \theta, m)$-sonvex and $-g_i$ is $(\varphi, \eta, \tilde{\rho}_i, \theta, m)$-sonvex at x^*, φ is superlinear, and $\varphi(a) \geq 0 \Rightarrow a \geq 0$;

(ii) for each $j \in J_+ \equiv J_+(v^*)$, G_j is $(\hat{\varphi}_j, \eta, \hat{\rho}_j, \theta, m)$-quasisonvex at x^*, $\hat{\varphi}_j$ is increasing, and $\hat{\varphi}_j(0) = 0$;

(iii) for each $k \in K_* \equiv K_*(w^*)$, $\xi \to \mathcal{D}_k(\xi, w^*)$ is $(\breve{\varphi}_k, \eta, \breve{\rho}_k, \theta, m)$-quasisonvex at x^* and $\breve{\varphi}_k(0) = 0$;

(iv) $\rho^*(x, x^*) + \sum_{j \in J_+} v_j^* \hat{\rho}_j(x, x^*) + \sum_{k \in K_*} \breve{\rho}_k(x, x^*) \geq 0$ for all $x \in \mathbb{F}$, where $\rho^*(x, x^*) = \sum_{i \in I_+} u_i^* [D(x^*, u^*) \bar{\rho}_i(x, x^*) + N(x^*, u^*) \tilde{\rho}_i(x, x^*)]$;

(b) (i) for each $i \in I_+,\ f_i$ is $(\varphi, \eta, \bar{\rho}_i, \theta, m)$-sonvex and $-g_i$ is $(\varphi, \eta, \tilde{\rho}_i, \theta, m)$-sonvex at $x^*,\ \varphi$ is superlinear, and $\varphi(a) \geq 0 \Rightarrow a \geq 0$;

(ii) $\xi \to \mathcal{C}(\xi, v^*)$ is $(\hat{\varphi}, \eta, \hat{\rho}, \theta, m)$-quasisonvex at $x^*,\ \hat{\varphi}$ is increasing, and $\hat{\varphi}(0) = 0$;

(iii) for each $k \in K_*,\ \xi \to \mathcal{D}_k(\xi, w^*)$ is $(\breve{\varphi}_k, \eta, \breve{\rho}_k, \theta, m)$-quasisonvex at x^* and $\breve{\varphi}_k(0) = 0$;

(iv) $\rho^*(x, x^*) + \hat{\rho}(x, x^*) + \sum_{k \in K_*} \breve{\rho}_k(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(c) (i) for each $i \in I_+,\ f_i$ is $(\varphi, \eta, \bar{\rho}_i, \theta, m)$-sonvex and $-g_i$ is $(\varphi, \eta, \tilde{\rho}_i, \theta, m)$-sonvex at $x^*,\ \varphi$ is superlinear, and $\varphi(a) \geq 0 \Rightarrow a \geq 0$;

(ii) for each $j \in J_+,\ G_j$ is $(\hat{\varphi}_j, \eta, \hat{\rho}_j, \theta, m)$-quasisonvex at $x^*,\ \hat{\varphi}_j$ is increasing, and $\hat{\varphi}_j(0) = 0$;

(iii) $\xi \to \mathcal{D}(\xi, w^*)$ is $(\breve{\varphi}, \eta, \breve{\rho}, \theta, m)$-quasisonvex at x^* and $\breve{\varphi}(0) = 0$;

(iv) $\rho^*(x, x^*) + \sum_{j \in J_+} v_j^* \hat{\rho}_j(x, x^*) + \breve{\rho}(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(d) (i) for each $i \in I_+,\ f_i$ is $(\varphi, \eta, \bar{\rho}_i, \theta, m)$-sonvex and $-g_i$ is $(\varphi, \eta, \tilde{\rho}_i, \theta, m)$-sonvex at $x^*,\ \varphi$ is superlinear, and $\varphi(a) \geq 0 \Rightarrow a \geq 0$;

(ii) $\xi \to \mathcal{C}(\xi, v^*)$ is $(\hat{\varphi}, \eta, \hat{\rho}, \theta, m)$-quasisonvex at $x^*,\ \hat{\varphi}$ is increasing, and $\hat{\varphi}(0) = 0$;

(iii) $\xi \to \mathcal{D}(\xi, w^*)$ is $(\breve{\varphi}, \eta, \breve{\rho}, \theta, m)$-quasisonvex at x^* and $\breve{\varphi}(0) = 0$;

(iv) $\rho^*(x, x^*) + \hat{\rho}(x, x^*) + \breve{\rho}(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(e) (i) for each $i \in I_+,\ f_i$ is $(\varphi, \eta, \bar{\rho}_i, \theta, m)$-sonvex and $-g_i$ is $(\varphi, \eta, \tilde{\rho}_i, \theta, m)$-sonvex at $x^*,\ \varphi$ is superlinear, and $\varphi(a) \geq 0 \Rightarrow a \geq 0$;

(ii) $\xi \to \mathcal{G}(\xi, v^*, w^*)$ is $(\hat{\varphi}, \eta, \hat{\rho}, \theta, m)$-quasisonvex at $x^*,\ \hat{\varphi}$ is increasing, and $\hat{\varphi}(0) = 0$;

(iii) $\rho^*(x, x^*) + \hat{\rho}(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(f) the Lagrangian-type function

$$\xi \to L(\xi, x^*, u^*, v^*, w^*) = \sum_{i=1}^{p} u_i^* [D(x^*, u^*) f_i(\xi) - N(x^*, u^*) g_i(\xi)] + \sum_{j=1}^{q} v_j^* G_j(\xi) + \sum_{k=1}^{r} w_k^* H_k(\xi)$$

is $(\varphi, \eta, \rho, \theta, m)$-pseudosonvex at $x^*,\ \rho(x, x^*) \geq 0$ for all $x \in \mathbb{F}$, and $\varphi(a) \geq 0 \Rightarrow a \geq 0$.

Then x^* is an optimal solution of (P).

Proof. Let x be an arbitrary feasible solution of (P).
(a) : Using the hypotheses specified in (i), we have for each $i \in I_+$,

$$\varphi\big(f_i(x) - f_i(x^*)\big) \geq \langle \nabla f_i(x^*), \eta(x, x^*) \rangle + \frac{1}{2} \langle z^*, \nabla^2 f_i(x^*) z^* \rangle + \bar{\rho}_i(x, x^*) \|\theta(x, x^*)\|^m$$

and

$$\varphi\big(- g_i(x) + g_i(x^*)\big) \geq \langle -\nabla g_i(x^*), \eta(x,x^*)\rangle - \frac{1}{2}\langle z^*, \nabla^2 g_i(x^*)z^*\rangle + \tilde{\rho}_i(x,x^*)\|\theta(x,x^*)\|^m.$$

Inasmuch as $D(x^*,u^*) > 0,\ N(x^*,u^*) \geq 0,\ u^* \geq 0,\ \sum_{i=1}^p u_i^* = 1$, and φ is superlinear, we deduce from the above inequalities that

$$\begin{aligned}&\varphi\Big(\sum_{i=1}^p u_i^*[D(x^*,u^*)f_i(x) - N(x^*,u^*)g_i(x)] - \sum_{i=1}^p u_i^*[D(x^*,u^*)f_i(x^*) - N(x^*,u^*)g_i(x^*)]\Big)\\ &\geq \Big\langle \sum_{i=1}^p u_i^*[D(x^*,u^*)\nabla f_i(x^*) - N(x^*,u^*)\nabla g_i(x^*)], \eta(x,x^*)\Big\rangle + \frac{1}{2}\Big\langle z^*, \sum_{i=1}^p u_i^*[D(x^*,u^*)\nabla^2 f_i(x^*)\\ &\quad - N(x^*,u^*)\nabla^2 g_i(x^*)]z^*\Big\rangle + \sum_{i\in I_+} u_i^*[D(x^*,u^*)\bar{\rho}_i(x,x^*) + N(x^*,u^*)\tilde{\rho}_i(x,x^*)]\|\theta(x,x^*)\|^m. \quad (2.6)\end{aligned}$$

Since $x \in \mathbb{F}$ and (2.5) holds, it follows from the properties of the functions $\hat{\varphi}_j$ that for each $j \in J_+,\ \hat{\varphi}_j\big(G_j(x) - G_j(x^*)\big) \leq 0$ which in view of (ii) implies that

$$\langle \nabla G_j(x^*), \eta(x,x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 G_j(x^*)z\rangle \leq -\hat{\rho}_j(x,x^*)\|\theta(x,x^*)\|^m.$$

As $v_j^* \geq 0$ for each $j \in \underline{q}$ and $v_j^* = 0$ for each $j \in \underline{q}\backslash J_+$ (complement of J_+ relative to $\underline{q}$), the above inequalities yield

$$\Big\langle \sum_{j=1}^q v_j^* \nabla G_j(x^*), \eta(x,x^*)\Big\rangle + \frac{1}{2}\Big\langle z^*, \sum_{j=1}^q v_j^* \nabla^2 G_j(x^*)z^*\Big\rangle \leq -\sum_{j\in J_+} v_j^*\hat{\rho}_j(x,x^*)\|\theta(x,x^*)\|^m. \quad (2.7)$$

In a similar manner, we can show that (iii) leads to the following inequality:

$$\Big\langle \sum_{k=1}^r w_k^* \nabla H_k(x^*), \eta(x,x^*)\Big\rangle + \frac{1}{2}\Big\langle z^*, \sum_{k=1}^r w_k^* \nabla^2 H_k(x^*)z^*\Big\rangle \leq -\sum_{k\in K_*} w_k^*\breve{\rho}_k(x,x^*)\|\theta(x,x^*)\|^m. \quad (2.8)$$

Now using (2.1), (2.2), and (2.6) - (2.8), we see that

$$\begin{aligned}&\varphi\Big(\sum_{i=1}^p u_i^*[D(x^*,u^*)f_i(x) - N(x^*,u^*)g_i(x)] - \sum_{i=1}^p u_i^*[D(x^*,u^*)f_i(x^*) - N(x^*,u^*)g_i(x^*)]\Big)\\ &\geq -\Big[\Big\langle \sum_{j=1}^q v_j^* \nabla G_j(x^*), \eta(x,x^*)\Big\rangle + \frac{1}{2}\Big\langle z^*, \sum_{j=1}^q v_j^* \nabla^2 G_j(x^*)z^*\Big\rangle\\ &\quad + \Big\langle \sum_{k=1}^r w_k^* \nabla H_k(x^*), \eta(x,x^*)\Big\rangle + \frac{1}{2}\Big\langle z^*, \sum_{k=1}^r w_k^* \nabla^2 H_k(x^*)z^*\Big\rangle\Big]\\ &\quad + \sum_{i\in I_+} u_i^*[D(x^*,u^*)\bar{\rho}_i(x,x^*) + N(x^*,u^*)\tilde{\rho}_i(x,x^*)]\|\theta(x,x^*)\|^m \ \ \text{(by (2.1), (2.2), and (2.6))}\end{aligned}$$

$$\geq \Big\{ \sum_{i\in I_+} u_i^*[D(x^*,u^*)\bar{\rho}_i(x,x^*) + N(x^*,u^*)\tilde{\rho}_i(x,x^*)] + \sum_{j\in J_+} v_j^*\hat{\rho}_j(x,x^*)$$
$$+ \sum_{k\in K_*} w_k^*\breve{\rho}_k(x,x^*)\Big\}\|\theta(x,x^*)\|^m \ \text{(by (2.7) and (2.8))}$$
$$\geq 0 \ \text{(by (iv))}.$$

But $\varphi(a) \geq 0 \Rightarrow a \geq 0$, and hence we have

$$\sum_{i=1}^{p} u_i^*[D(x^*,u^*)f_i(x) - N(x^*,u^*)g_i(x)] \geq \sum_{i=1}^{p} u_i^*[D(x^*,u^*)f_i(x^*) - N(x^*,u^*)g_i(x^*)] = 0, \quad (2.9)$$

where the equality follows from (2.3). Now using (2.4), (2.9) and Lemma 2.1, we see that

$$\varphi(x^*) = \frac{N(x^*,u^*)}{D(x^*,u^*)} \leq \frac{\sum_{i=1}^{p} u_i^* f_i(x)}{\sum_{i=1}^{p} u_i^* g_i(x)} \leq \max_{u\in U} \frac{\sum_{i=1}^{p} u_i f_i(x)}{\sum_{i=1}^{p} u_i g_i(x)} = \varphi(x).$$

Since $x \in \mathbb{F}$ was arbitrary, we conclude from this inequality that x^* is an optimal solution of (P).

(b) : As shown in part (a), for each $j \in J_+$, we have $G_j(x) - G_j(x^*) \leq 0$, and hence using the properties of the function $\hat{\varphi}$, we get

$$\hat{\varphi}\Big(\sum_{j=1}^{q} v_j^* G_j(x) - \sum_{j=1}^{q} v_j^* G_j(x^*)\Big) \leq 0,$$

which in view of (ii) implies that

$$\Big\langle \sum_{j=1}^{q} v_j^* \nabla G_j(x^*), \eta(x,x^*)\Big\rangle + \frac{1}{2}\Big\langle z^*, \sum_{j=1}^{q} v_j^* \nabla^2 G_j(x^*) z^*\Big\rangle \leq -\hat{\rho}(x,x^*)\|\theta(x,x^*)\|^m.$$

Now proceeding as in the proof of part (a) and using this inequality instead of (2.7), we arrive at (2.9), which leads to the desired conclusion that x^* is an optimal solution of (P).

(c) - (e) : The proofs are similar to those of parts (a) and (b).

(f) : Since $\rho(x,x^*) \geq 0$, (2.1) and (2.2) yield

$$\langle \nabla L(x^*,x^*,u^*,v^*,w^*), \eta(x,x^*)\rangle + \frac{1}{2}\langle z^*, \nabla^2 L(x^*,x^*,u^*,v^*,w^*) z^*\rangle$$
$$\geq \ 0 \ \geq -\rho(x,x^*)\|\theta(x,x^*)\|^m,$$

which in view of our $(\varphi,\eta,\rho,\theta,m)$-pseudosonvexity assumption implies that

$$\varphi\big(L(x,x^*,u^*,v^*,w^*) - L(x^*,x^*,u^*,v^*,w^*)\big) \geq 0.$$

But $\varphi(a) \geq 0 \Rightarrow a \geq 0$ and hence we have

$$L(x,x^*,u^*,v^*,w^*) \geq L(x^*,x^*,u^*,v^*,w^*).$$

Because $x, x^* \in \mathbb{F},\ v^* \geq 0$, and (2.3) and (2.5) hold, the right-hand side of the above inequality is equal to zero, and so we get

$$\sum_{i=1}^{p} u_i^*[D(x^*, u^*)f_i(x) - N(x^*, u^*)g_i(x)] \geq 0,$$

which is precisely (2.9). As seen in the proof of part (a), this inequality leads to the desired conclusion that x^* is an optimal solution of (P). Q.E.D.

In Theorem 2.1, separate $(\varphi, \eta, \rho, \theta, m)$-sonvexity assumptions were imposed on the functions f_i and $-g_i,\ i \in \underline{p}$. It is possible to establish a great variety of additional sufficient optimality results in which various generalized $(\varphi, \eta, \rho, \theta, m)$-sonvexity requirements are placed on certain combinations of these functions. In the remainder of this paper, we shall discuss a series of sufficiency theorems in which appropriate generalized $(\varphi, \eta, \rho, \theta, m)$-sonvexity assumptions will be imposed on the functions $\xi \to \mathcal{E}_i(\xi, y, u),\ i \in \underline{p},\ \xi \to \mathcal{E}(\xi, y, u),\ G_j,\ j \in \underline{q},\ \xi \to \mathcal{C}(\xi, v),\ \xi \to \mathcal{D}_k(\xi, w),\ k \in \underline{r},\ \xi \to \mathcal{D}(\xi, w)$, and $\xi \to \mathcal{G}(\xi, v, w)$.

Theorem 2.3. Let $x^* \in \mathbb{F}$ and assume that the functions $f_i,\ g_i,\ i \in \underline{p},\ G_j,\ j \in \underline{q}$, and $H_k,\ k \in \underline{r}$, are twice differentiable at x^*, and that for each $z^* \in C(x^*)$, there exist $u^* \in U,\ v^* \in \mathbb{R}^q_+$, and $w^* \in \mathbb{R}^r$ such that (2.1) - (2.5) hold. Assume, furthermore, that any one of the following five sets of hypotheses is satisfied:

(a) (i) $\xi \to \mathcal{E}(\xi, x^*, u^*)$ is $(\bar{\varphi}, \eta, \bar{\rho}, \theta, m)$-pseudosonvex at x^*, and $\bar{\varphi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) for each $j \in J_+ \equiv J(v^*),\ G_j$ is $(\hat{\varphi}_j, \eta, \hat{\rho}_j, \theta, m)$-quasisonvex at $x^*,\ \hat{\varphi}_j$ is increasing, and $\hat{\varphi}_j(0) = 0$;

(iii) for each $k \in K_* \equiv K(w^*),\ \xi \to \mathcal{D}_k(\xi, w^*)$ is $(\breve{\varphi}_k, \eta, \breve{\rho}_k, \theta, m)$-quasisonvex at x^*, and $\breve{\varphi}_k(0) = 0$;

(iv) $\bar{\rho}(x, x^*) + \sum_{j \in J_+} v_j^* \hat{\rho}_j(x, x^*) + \sum_{k \in K_*} \breve{\rho}_k(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(b) (i) $\xi \to \mathcal{E}(\xi, x^*, u^*, \lambda^*)$ is $(\bar{\varphi}, \eta, \bar{\rho}, \theta, m)$-pseudosonvex at x^*, and $\bar{\varphi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) $\xi \to \mathcal{C}(\xi, v^*)$ is $(\hat{\varphi}, \eta, \hat{\rho}, \theta, m)$-quasisonvex at $x^*,\ \hat{\varphi}$ is increasing, and $\hat{\varphi}(0) = 0$;

(iii) for each $k \in K_*,\ \xi \to \mathcal{D}_k(\xi, w^*)$ is $(\breve{\varphi}_k, \eta, \breve{\rho}_k, \theta, m)$-quasisonvex at x^*, and $\breve{\varphi}_k(0) = 0$;

(iv) $\bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) + \sum_{k \in K_*} \breve{\rho}_k(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(c) (i) $\xi \to \mathcal{E}(\xi, x^*, u^*, \lambda^*)$ is $(\bar{\varphi}, \eta, \bar{\rho}, \theta, m)$-pseudosonvex at x^*, and $\bar{\varphi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) for each $j \in J_+,\ G_j$ is $(\hat{\varphi}_m, \eta, \hat{\rho}_j, \theta, m)$-quasisonvex at $x^*,\ \hat{\varphi}_j$ is increasing, and $\hat{\varphi}_j(0) = 0$;

(iii) $\xi \to \mathcal{D}(\xi, w^*)$ is $(\breve{\varphi}, \eta, \breve{\rho}, \theta, m)$-quasisonvex at x^*, and $\breve{\varphi}(0) = 0$;

(iv) $\bar{\rho}(x, x^*) + \sum_{j \in J_+} v_j^* \hat{\rho}_j(x, x^*) + \breve{\rho}(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(d) (i) $\xi \to \mathcal{E}(\xi, x^*, u^*, \lambda^*)$ is $(\bar{\varphi}, \eta, \bar{\rho}, \theta, m)$-pseudosonvex at x^*, and $\bar{\varphi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) $\xi \to \mathcal{C}(\xi, v^*)$ is $(\hat{\varphi}, \eta, \hat{\rho}, \theta, m)$-quasisonvex at $x^*,\ \hat{\varphi}$ is increasing, and $\hat{\varphi}(0) = 0$;

(iii) $\xi \to \mathcal{D}(\xi, w^*)$ is $(\breve{\varphi}, \eta, \breve{\rho}, \theta, m)$-quasisonvex at x^*, and $\breve{\varphi}(0) = 0$;

(iv) $\bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) + \breve{\rho}(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(e) (i) $\xi \to \mathcal{E}(\xi, x^*, u^*, \lambda^*)$ is $(\bar{\varphi}, \eta, \bar{\rho}, \theta, m)$-pseudosonvex at x^*, and $\bar{\varphi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) $\xi \to \mathcal{G}(\xi, v^*, w^*)$ is $(\hat{\varphi}, \eta, \hat{\rho}, \theta, m)$-quasisonvex at x^*, $\hat{\varphi}$ is increasing, and $\hat{\varphi}(0) = 0$;

(iii) $\bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) \geq 0$ for all $x \in \mathbb{F}$.

Then x^* is an optimal solution of (P).

Proof. Let x be an arbitrary feasible solution of (P).

(a) : In view of our assumptions specified in (ii) and (iii), (2.7) and (2.8) remain valid for the present case. From (2.1), (2.2), (2.7), (2.8), and (iv) we deduce that

$$\Big\langle \sum_{i=1}^{p} u_i^*[D(x^*, u^*)\nabla f_i(x^*) - N(x^*, u^*)\nabla g_i(x^*)], \eta(x, x^*)\Big\rangle + \Big\langle z^*, \sum_{i=1}^{p} u_i^*[D(x^*, u^*)\nabla^2 f_i(x^*) - N(x^*, u^*)\nabla^2 g_i(x^*)]z^*\Big\rangle \geq \Big[\sum_{j\in J_+} v_j^* \hat{\rho}_j(x, x^*) + \sum_{k\in K_*} \breve{\rho}_k(x, x^*)\Big]\|\theta(x, x^*)\|^m \geq -\bar{\rho}(x, x^*)\|\theta(x, x^*)\|^m,$$

which in view of (i) implies that

$$\bar{\varphi}\big(\mathcal{E}(x, x^*, u^*) - \mathcal{E}(x^*, x^*, u^*)\big) \geq 0.$$

Because of the properties of the function $\bar{\varphi}$, the last inequality yields

$$\mathcal{E}(x, x^*, u^*) \geqq \mathcal{E}(x^*, x^*, u^*) = 0,$$

where the equality follows from (2.3). As shown in the proof of Theorem 2.1, this inequality leads to the conclusion that x^* is an optimal solution of (P).

(b) - (e) : The proofs are similar to that of part (a). Q.E.D.

Theorem 2.4. Let $x^* \in \mathbb{F}$ and assume that the functions $f_i,\ g_i,\ i \in \underline{p},\ G_j,\ j \in \underline{q}$, and $H_k,\ k \in \underline{r}$, are twice differentiable at x^*, and that there exist $u^* \in U,\ v^* \in \mathbb{R}_+^q$, and $w^* \in \mathbb{R}^r$ such that (2.1) - (2.5) hold. Assume, furthermore, that any one of the following five sets of hypotheses is satisfied:

(a) (i) $\xi \to \mathcal{E}(\xi, x^*, u^*)$ is prestrictly $(\bar{\varphi}, \eta, \bar{\rho}, \theta, m)$-quasisonvex at x^*, and $\bar{\varphi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) for each $j \in J_+ \equiv J_+(v^*)$, G_j is $(\hat{\varphi}_j, \eta, \hat{\rho}_j, \theta, m)$-quasisonvex at x^*, $\hat{\varphi}_j$ is increasing, and $\hat{\varphi}_j(0) = 0$;

(iii) for each $k \in K_* \equiv K(w^*)$, $\xi \to \mathcal{D}_k(\xi, w^*)$ is $(\breve{\varphi}_k, \eta, \breve{\rho}_k, \theta, m)$-quasisonvex at x^*, and $\breve{\varphi}_k(0) = 0$;

(iv) $\bar{\rho}(x, x^*) + \sum_{j\in J_+} v_j^* \hat{\rho}_j(x, x^*) + \sum_{k\in K_*} \breve{\rho}_k(x, x^*) > 0$ for all $x \in \mathbb{F}$;

(b) (i) $\xi \to \mathcal{E}(\xi, x^*, u^*)$ is prestrictly $(\bar{\varphi}, \eta, \bar{\rho}, \theta, m)$-quasisonvex at x^*, and $\bar{\varphi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) $\xi \to \mathcal{C}(\xi, v^*)$ is $(\hat{\varphi}, \eta, \hat{\rho}, \theta, m)$-quasisonvex at x^*, $\hat{\varphi}$ is increasing, and $\hat{\varphi}(0) = 0$;

(iii) for each $k \in K_*$, $\xi \to \mathcal{D}_k(\xi, w^*)$ is $(\breve{\varphi}_k, \eta, \breve{\rho}_k, \theta, m)$-quasisonvex at x^*, and $\breve{\varphi}_k(0) = 0$;

(iv) $\bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) + \sum_{k \in K_*} \breve{\rho}_k(x, x^*) > 0$ for all $x \in \mathbb{F}$;

(c) (i) $\xi \to \mathcal{E}(\xi, x^*, u^*)$ is prestrictly $(\bar{\varphi}, \eta, \bar{\rho}, \theta, m)$-quasisonvex at x^*, and $\bar{\varphi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) for each $j \in J_+$, G_j is $(\hat{\varphi}_j, \eta, \hat{\rho}_j, \theta, m)$-quasisonvex at x^*, $\hat{\varphi}_j$ is increasing, and $\hat{\varphi}_j(0) = 0$;

(iii) $\xi \to \mathcal{D}(\xi, w^*)$ is $(\breve{\varphi}, \eta, \breve{\rho}, \theta, m)$-quasisonvex at x^*, and $\breve{\varphi}(0) = 0$;

(iv) $\bar{\rho}(x, x^*) + \sum_{j \in J_+} v_j^* \hat{\rho}_j(x, x^*) + \breve{\rho}(x, x^*) > 0$ for all $x \in \mathbb{F}$;

(d) (i) $\xi \to \mathcal{E}(\xi, x^*, u^*)$ is prestrictly $(\bar{\varphi}, \eta, \bar{\rho}, \theta, m)$-quasisonvex at x^*, and $\bar{\varphi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) $\xi \to \mathcal{C}(\xi, w^*)$ is $(\hat{\varphi}, \eta, \hat{\rho}, \theta, m)$-quasisonvex at x^*, $\hat{\varphi}$ is increasing, and $\hat{\varphi}(0) = 0$;

(iii) $\xi \to \mathcal{D}(\xi, w^*)$ is $(\breve{\varphi}, \eta, \breve{\rho}, \theta, m)$-quasisonvex at x^*, and $\breve{\varphi}(0) = 0$;

(iv) $\bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) + \breve{\rho}(x, x^*) > 0$ for all $x \in \mathbb{F}$;

(e) (i) $\xi \to \mathcal{E}(\xi, x^*, u^*)$ is prestrictly $(\bar{\varphi}, \eta, \bar{\rho}, \theta, m)$-quasisonvex at x^*, and $\bar{\varphi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) $\xi \to \mathcal{G}(\xi, v^*, w^*)$ is $(\hat{\varphi}, \eta, \hat{\rho}, \theta, m)$-quasisonvex at x^*, $\hat{\varphi}$ is increasing, and $\hat{\varphi}(0) = 0$;

(iii) $\bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) > 0$ for all $x \in \mathbb{F}$.

Then x^* is an optimal solution of (P).

Proof. Let x be an arbitrary feasible solution of (P).
(a) : Because of our assumptions specified in (ii) and (iii), (2.7) and (2.8) remain valid for the present case. From (2.1), (2.2), (2.7), (2.8), and (iv) we deduce that

$$\Big\langle \sum_{i=1}^{p} u_i^* [D(x^*, u^*) \nabla f_i(x^*) - N(x^*, u^*) \nabla g_i(x^*)], \eta(x, x^*) \Big\rangle + \Big\langle z^*, \sum_{i=1}^{p} u_i^* [D(x^*, u^*) \nabla^2 f_i(x^*) - N(x^*, u^*) \nabla^2 g_i(x^*)] z^* \Big\rangle \geq \Big[\sum_{j \in J_+} v_j^* \hat{\rho}_j(x, x^*) + \sum_{k \in K_*} \breve{\rho}_k(x, x^*) \Big] \|\theta(x, x^*)\|^m > -\bar{\rho}(x, x^*) \|\theta(x, x^*)\|^m,$$

which in view of (i) implies that

$$\bar{\varphi}\big(\mathcal{E}(x, x^*, u^*) - \mathcal{E}(x^*, x^*, u^*)\big) \geq 0.$$

Because of the properties of the function $\bar{\varphi}$, the last inequality yields

$$\mathcal{E}(x, x^*, u^*) \geq \mathcal{E}(x^*, x^*, u^*) = 0,$$

where the equality follows from (2.3). As shown in the proof of Theorem 2.1, this inequality leads to the conclusion that x^* is an optimal solution of (P).
(b) - (e) : The proofs are similar to that of part (a). Q.E.D.

Theorem 2.5. Let $x^* \in \mathbb{F}$ and assume that the functions $f_i,\ g_i,\ i \in \underline{p},\ G_j,\ j \in \underline{q}$, and $H_k,\ k \in \underline{r}$, are twice differentiable at x^*, and that there exist $u^* \in U,\ v^* \in \mathbb{R}^q_+$, and $w^* \in \mathbb{R}^r$ such that (2.1) - (2.5) hold. Assume, furthermore, that any one of the following seven sets of hypotheses is satisfied:

(a) (i) $\xi \to \mathcal{E}(\xi, x^*, u^*)$ is prestrictly $(\bar{\varphi}, \eta, \bar{\rho}, \theta, m)$-quasisonvex at x^* and $\bar{\varphi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) for each $j \in J_+ \equiv J_+(v^*)$, G_j is strictly $(\hat{\varphi}_j, \eta, \hat{\rho}_j, \theta, m)$-pseudosonvex at x^*, $\hat{\varphi}_j$ is increasing, and $\hat{\varphi}_j(0) = 0$;

(iii) for each $k \in K_* \equiv K_*(w^*)$, $\xi \to \mathcal{D}_k(\xi, w^*)$ is $(\breve{\varphi}_k, \eta, \breve{\rho}_k, \theta, m)$-quasisonvex at x^*, and $\breve{\varphi}_k(0) = 0$;

(iv) $\bar{\rho}(x, x^*) + \sum_{j \in J_+} v_j^* \hat{\rho}_j(x, x^*) + \sum_{k \in K_*} \breve{\rho}_k(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(b) (i) $\xi \to \mathcal{E}(\xi, x^*, u^*)$ is prestrictly $(\bar{\varphi}, \eta, \bar{\rho}, \theta, m)$-quasisonvex at x^*, and $\bar{\varphi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) $\xi \to \mathcal{C}(\xi, v^*)$ is strictly $(\hat{\varphi}, \eta, \hat{\rho}, \theta, m)$-pseudosonvex at x^*, $\hat{\varphi}$ is increasing, and $\hat{\varphi}(0) = 0$;

(iii) for each $k \in K_*$, $\xi \to \mathcal{D}_k(\xi, w^*)$ is $(\breve{\varphi}_k, \eta, \breve{\rho}_k, \theta, m)$-quasisonvex at x^*, and $\breve{\varphi}_k(0) = 0$;

(iv) $\bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) + \sum_{k \in K_*} \breve{\rho}_k(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(c) (i) $\xi \to \mathcal{E}(\xi, x^*, u^*)$ is prestrictly $(\bar{\varphi}, \eta, \bar{\rho}, \theta, m)$-quasisonvex at x^* and $\bar{\varphi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) for each $j \in J_+$, G_j is $(\hat{\varphi}_j, \eta, \hat{\rho}_j, \theta, m)$-quasisonvex at x^*, $\hat{\varphi}_j$ is increasing, and $\hat{\varphi}_j(0) = 0$;

(iii) for each $k \in K_*$, $\xi \to \mathcal{D}_k(\xi, w^*)$ is strictly $(\breve{\varphi}_k, \eta, \breve{\rho}_k, \theta, m)$-pseudosonvex at x^*, and $\breve{\varphi}_k(0) = 0$;

(iv) $\bar{\rho}(x, x^*) + \sum_{j \in J_+} v_j^* \hat{\rho}_j(x, x^*) + \sum_{k \in K_*} \breve{\rho}_m(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(d) (i) $\xi \to \mathcal{E}(\xi, x^*, u^*)$ is prestrictly $(\bar{\varphi}, \eta, \bar{\rho}, \theta, m)$-quasisonvex at x^* and $\bar{\varphi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) for each $j \in J_+$, $\xi \to G_j$ is $(\hat{\varphi}_j, \eta, \hat{\rho}_j, \theta, m)$-quasisonvex at x^*, $\hat{\varphi}_j$ is increasing, and $\hat{\varphi}_j(0) = 0$;

(iii) $\xi \to \mathcal{D}(\xi, w^*)$ is strictly $(\breve{\varphi}, \eta, \breve{\rho}, \theta, m)$-pseudosonvex at x^*, and $\breve{\varphi}(0) = 0$;

(iv) $\bar{\rho}(x, x^*) + \sum_{j \in J_+} v_j^* \hat{\rho}_j(x, x^*) + \breve{\rho}(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(e) (i) $\xi \to \mathcal{E}(\xi, x^*, u^*)$ is prestrictly $(\bar{\varphi}, \eta, \bar{\rho}, \theta, m)$-quasisonvex at x^* and $\bar{\varphi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) $\xi \to \mathcal{C}(\xi, v^*)$ is strictly $(\hat{\varphi}, \eta, \hat{\rho}, \theta, m)$-pseudosonvex at x^*, $\hat{\varphi}$ is increasing, and $\hat{\varphi}(0) = 0$;

(iii) $\xi \to \mathcal{D}(\xi, w^*)$ is $(\breve{\varphi}, \eta, \breve{\rho}, \theta, m)$-quasisonvex at x^*, and $\breve{\varphi}(0) = 0$;

(iv) $\bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) + \breve{\rho}(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(f) (i) $\xi \to \mathcal{E}(\xi, x^*, u^*)$ is prestrictly $(\bar{\varphi}, \eta, \bar{\rho}, \theta, m)$-quasisonvex at x^* and $\bar{\varphi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) $\xi \to \mathcal{C}(\xi, v^*)$ is $(\hat{\varphi}, \eta, \hat{\rho}, \theta, m)$-quasisonvex at x^*, $\hat{\varphi}$ is increasing, and $\hat{\varphi}(0) = 0$;

(iii) $\xi \to \mathcal{D}(\xi, w^*)$ is strictly $(\breve{\varphi}, \eta, \breve{\rho}, \theta, m)$-pseudosonvex at x^*, and $\breve{\varphi}(0) = 0$;

(iv) $\bar{\rho}(x,x^*)+\hat{\rho}(x,x^*)+\breve{\rho}(x,x^*)\geq 0$ for all $x\in\mathbb{F}$;

(g) (i) $\xi\to\mathcal{E}(\xi,x^*,u^*)$ is prestrictly $(\bar{\varphi},\eta,\bar{\rho},\theta,m)$-quasisonvex at x^* and $\bar{\varphi}(a)\geq 0\Rightarrow a\geq 0$;

(ii) $\xi\to\mathcal{G}(\xi,v^*,w^*)$ is strictly $(\hat{\varphi},\eta,\hat{\rho},\theta,m)$-pseudosonvex at x^*, $\hat{\varphi}$ is increasing, and $\hat{\varphi}(0)=0$;

(iii) $\bar{\rho}(x,x^*)+\hat{\rho}(x,x^*)\geq 0$ for all $x\in\mathbb{F}$.

Then x^* is an optimal solution of (P).

Proof. (a) : Let $x\in\mathbb{F}$ be an arbitrary feasible solution of (P). Since for each $j\in J_+$, $G_j(x)-G_j(x^*)\leq 0$ and hence $\hat{\varphi}_j\big(G_j(x)-G_j(x^*)\big)\leq 0$, (ii) implies that

$$\langle\nabla G_j(x^*),\eta(x,x^*)\rangle+\frac{1}{2}\langle z,\nabla^2 G_j(x^*)z\rangle<-\hat{\rho}_j(x,x^*)\|\theta(x,x^*)\|^m.$$

As $v_j^*\geq 0$ for each $j\in\underline{q}$ and $v_j^*=0$ for each $j\in\underline{q}\backslash J_+$, the above inequalities yield

$$\Big\langle\sum_{j=1}^{q}v_j^*\nabla G_j(x^*),\eta(x,x^*)\Big\rangle+\frac{1}{2}\Big\langle z^*,\sum_{j=1}^{q}v_j^*\nabla^2 G_j(x^*)z^*\Big\rangle<-\sum_{j\in J_+}v_j^*\hat{\rho}_j(x,x^*)\|\theta(x,x^*)\|^m.$$

Now combining this inequality with (2.1), (2.2), and (2.8) (which is valid for the present case because of (iii)), and using (iv), we obtain

$$\begin{aligned}\Big\langle\sum_{i=1}^{p}u_i^*[D(x^*,u^*)\nabla f_i(x^*)-N(x^*,u^*)\nabla g_i(x^*)],\eta(x,x^*)\Big\rangle+\frac{1}{2}\Big\langle z^*,\sum_{i=1}^{p}u_i^*[D(x^*,u^*)\nabla^2 f_i(x^*)\\ -N(x^*,u^*)\nabla^2 g_i(x^*)]z^*\Big\rangle\geq\Big[\sum_{j\in J_+}v_j^*\hat{\rho}_j(x,x^*)+\sum_{k\in K_*}\breve{\rho}_k(x,x^*)\Big]\|\theta(x,x^*)\|^m\\ >-\bar{\rho}(x,x^*)\|\theta(x,x^*)\|^m,\end{aligned}$$

which in view of (i) implies that

$$\bar{\varphi}\big(\mathcal{E}(x,x^*,u^*)-\mathcal{E}(x^*,x^*,u^*)\big)\geq 0.$$

The rest of the proof is identical to that of Theorem 2.1.

(b) - (g) : The proofs are similar to that of part (a). Q.E.D.

In Theorems 2.2 - 2.4, various generalized $(\varphi,\eta,\rho,\theta,m)$-sonvexity conditions were imposed on the function $\xi\to\mathcal{E}(\xi,x^*,u^*)$, which is the weighted sum of the functions $\xi\to\mathcal{E}_i(\xi,x^*,u^*)$, $i\in\underline{p}$. In the next few theorems, we shall assume that the individual functions $\xi\to\mathcal{E}_i(\xi,x^*,u^*)$, $i\in\underline{p}$, satisfy appropriate generalized $(\varphi,\eta,\rho,\theta,m)$-sonvexity hypotheses.

Theorem 2.6. Let $x^*\in\mathbb{F}$ and assume that the functions f_i, g_i, $i\in\underline{p}$, G_j, $j\in\underline{q}$, and H_k, $k\in\underline{r}$, are twice differentiable at x^*, and that for each $z^*\in C(x^*)$, there exist $u^*\in U$, $v^*\in\mathbb{R}_+^q$, and $w^*\in\mathbb{R}^r$ such that (2.1) - (2.5) hold. Assume, furthermore, that any one of the following five sets of hypotheses is satisfied:

(a) (i) for each $i \in I_+ \equiv I_+(u^*)$, $\xi \to \mathcal{E}_i(\xi, x^*, u^*)$ is prestrictly $(\bar{\varphi}_i, \eta, \bar{\rho}_i, \theta, m)$-quasisonvex at x^*, $\bar{\varphi}_i$ is strictly increasing, and $\bar{\varphi}_i(0) = 0$;

(ii) for each $j \in J_+ \equiv J_+(v^*)$, G_j is $(\hat{\varphi}_j, \eta, \hat{\rho}_j, \theta, m)$-quasisonvex at x^*, $\hat{\varphi}_j$ is increasing, and $\hat{\varphi}_j(0) = 0$;

(iii) for each $k \in K_* \equiv K_*(w^*)$, $\xi \to \mathcal{D}_k(\xi, w^*)$ is $(\breve{\varphi}_k, \eta, \breve{\rho}_k, \theta, m)$-quasisonvex at x^*, and $\breve{\varphi}_k(0) = 0$;

(iv) $\rho^\circ(x, x^*) + \sum_{j \in J_+} v_j^* \hat{\rho}_j(x, x^*) + \sum_{k \in K_*} \breve{\rho}_k(x, x^*) > 0$ for all $x \in \mathbb{F}$, where $\rho^\circ(x, x^*) = \sum_{i \in I_+} u_i^* \bar{\rho}_i(x, x^*)$;

(b) (i) for each $i \in I_+$, $\xi \to \mathcal{E}_i(\xi, x^*, u^*)$ is prestrictly $(\bar{\varphi}_i, \eta, \bar{\rho}_i, \theta, m)$-quasisonvex at x^*, $\bar{\varphi}_i$ is strictly increasing, and $\bar{\varphi}_i(0) = 0$;

(ii) $\xi \to \mathcal{C}(\xi, v^*)$ is $(\hat{\varphi}, \eta, \hat{\rho}, \theta, m)$-quasisonvex at x^*, $\hat{\varphi}$ is increasing, and $\hat{\varphi}(0) = 0$;

(iii) for each $k \in K_*$, $\xi \to \mathcal{D}_k(\xi, w^*)$ is $(\breve{\varphi}_k, \eta, \breve{\rho}_k, \theta, m)$-quasisonvex at x^*, and $\breve{\varphi}_k(0) = 0$;

(iv) $\rho^\circ(x, x^*) + \hat{\rho}(x, x^*) + \sum_{k \in K_*} \breve{\rho}_m(x, x^*) > 0$ for all $x \in \mathbb{F}$;

(c) (i) for each $i \in I_+$, $\xi \to \mathcal{E}_i(\xi, x^*, u^*)$ is prestrictly $(\bar{\varphi}_i, \eta, \bar{\rho}_i, \theta, m)$-quasisonvex at x^*, $\bar{\varphi}_i$ is strictly increasing, and $\bar{\varphi}_i(0) = 0$;

(ii) for each $j \in J_+$, G_j is $(\hat{\varphi}_j, \eta, \hat{\rho}_j, \theta, m)$-quasisonvex at x^*, $\hat{\varphi}_j$ is increasing, and $\hat{\varphi}_j(0) = 0$;

(iii) $\xi \to \mathcal{D}(\xi, w^*)$ is $(\breve{\varphi}, \eta, \breve{\rho}, \theta, m)$-quasisonvex at x^*, and $\breve{\varphi}(0) = 0$;

(iv) $\rho^\circ(x, x^*) + \sum_{j \in J_+} v_j^* \hat{\rho}_j(x, x^*) + \breve{\rho}(x, x^*) > 0$ for all $x \in \mathbb{F}$;

(d) (i) for each $i \in I_+$, $\xi \to \mathcal{E}_i(\xi, x^*, u^*)$ is prestrictly $(\bar{\varphi}_i, \eta, \bar{\rho}_i, \theta, m)$-quasisonvex at x^*, $\bar{\varphi}_i$ is strictly increasing, and $\bar{\varphi}_i(0) = 0$;

(ii) $\xi \to \mathcal{C}(\xi, v^*)$ is $(\hat{\varphi}, \eta, \hat{\rho}, \theta, m)$-quasisonvex at x^*, $\hat{\varphi}$ is increasing, and $\hat{\varphi}(0) = 0$;

(iii) $\xi \to \mathcal{D}(\xi, w^*)$ is $(\breve{\varphi}, \eta, \breve{\rho}, \theta, m)$-quasisonvex at x^*, and $\breve{\varphi}(0) = 0$;

(iv) $\rho^\circ(x, x^*) + \hat{\rho}(x, x^*) + \breve{\rho}(x, x^*) > 0$ for all $x \in \mathbb{F}$;

(e) (i) for each $i \in I_+$, $\xi \to \mathcal{E}_i(\xi, x^*, u^*)$ is prestrictly $(\bar{\varphi}_i, \eta, \bar{\rho}_i, \theta, m)$-quasisonvex at x^*, $\bar{\varphi}_i$ is strictly increasing, and $\bar{\varphi}_i(0) = 0$;

(ii) $\xi \to \mathcal{G}(\xi, v^*, w^*)$ is $(\hat{\varphi}, \eta, \hat{\rho}, \theta, m)$-quasisonvex at x^*, $\hat{\varphi}$ is increasing, and $\hat{\varphi}(0) = 0$;

(iii) $\rho^\circ(x, x^*) + \hat{\rho}(x, x^*) > 0$ for all $x \in \mathbb{F}$.

Then x^* is an optimal solution of (P).

Proof. (a) : Suppose that x^* is not an optimal solution of (P). This implies that there exists $\bar{x} \in \mathbb{F}$ such that for each $i \in \underline{p}$, $\mathcal{E}_i(\bar{x}, x^*, u^*) < 0$. Since $\mathcal{E}_i(x^*, x^*, u^*) = 0$ by (2.3), it follows that

$\mathcal{E}_i(\bar{x}, x^*, u^*) < \mathcal{E}_i(x^*, x^*, u^*)$, and hence for each $i \in I_+$, $\bar{\varphi}\big(\mathcal{E}_i(\bar{x}, x^*, u^*) - \mathcal{E}_i(x^*, x^*, u^*)\big) < 0$, which by virtue of (i) implies that

$$\langle D(x^*, u^*)\nabla f_i(x^*) - N(x^*, u^*)\nabla g_i(x^*), \eta(\bar{x}, x^*)\rangle + \frac{1}{2}\langle z^*, [D(x^*, u^*)\nabla^2 f_i(x^*) - N(x^*, u^*)\nabla^2 g_i(x^*)]z^*\rangle \leq -\bar{\rho}_i(\bar{x}, x^*)\|\theta(\bar{x}, x^*)\|^m.$$

Since $u^* \geq 0$ and $\sum_{i=1}^p u_i^* = 1$, the above inequalities yield

$$\Big\langle \sum_{i=1}^p u_i^*[D(x^*, u^*)\nabla f_i(x^*) - N(x^*, u^*)\nabla g_i(x^*)], \eta(\bar{x}, x^*)\Big\rangle + \frac{1}{2}\Big\langle z^*, \sum_{i=1}^p u_i^*[D(x^*, u^*)\nabla^2 f_i(x^*) - N(x^*, u^*)\nabla^2 g_i(x^*)]z^*\Big\rangle] \leq -\sum_{i\in I_+} u_i^*\bar{\rho}_i(x, x^*)\|\theta(x, x^*)\|^m. \quad (2.10)$$

In view of our assumptions set forth in (ii) and (iii), (2.7) and (2.8) hold. Now combining these inequalities with (2.1) and (2.2), and using (iv), we get

$$\Big\langle \sum_{i=1}^p u_i^*[D(x^*, u^*)\nabla f_i(x^*) - N(x^*, u^*)\nabla g_i(x^*)], \eta(\bar{x}, x^*)\Big\rangle + \frac{1}{2}\Big\langle z^*, \sum_{i=1}^p u_i^*[D(x^*, u^*)\nabla^2 f_i(x^*) - N(x^*, u^*)\nabla^2 g_i(x^*)]z^*\Big\rangle \geq \Big[\sum_{j\in J_+} v_j^*\hat{\rho}_j(x, x^*) + \sum_{k\in K_*} \breve{\rho}_k(x, x^*)\Big]\|\theta(x, x^*)\|^m > -\sum_{i\in I_+} u_i^*\bar{\rho}_i(x, x^*)\|\theta(x, x^*)\|^m,$$

which contradicts (2.10). Therefore, we conclude that x^* is an optimal solution of (P).

(b) - (e) : The proofs are similar to that of part (a). Q.E.D.

Theorem 2.7. Let $x^* \in \mathbb{F}$ and assume that the functions f_i, g_i, $i \in \underline{p}$, G_j, $j \in \underline{q}$, and H_k, $k \in \underline{r}$, are twice differentiable at x^*, and that for each $z^* \in C(x^*)$, there exist $u^* \in U$, $v^* \in \mathbb{R}_+^q$, and $w^* \in \mathbb{R}^r$ such that (2.1) - (2.5) hold. Assume, furthermore, that any one of the following seven sets of hypotheses is satisfied:

(a) (i) for each $i \in I_+ \equiv I_+(u^*)$, $\xi \to \mathcal{E}_i(\xi, x^*, u^*)$ is prestrictly $(\bar{\varphi}_i, \eta, \bar{\rho}_i, \theta, m)$-quasisonvex at x^*, $\bar{\varphi}_i$ is strictly increasing, and $\bar{\varphi}_i(0) = 0$;

(ii) for each $j \in J_+ \equiv J_+(v^*)$, G_j is strictly $(\hat{\varphi}_j, \eta, \hat{\rho}_j, \theta, m)$-pseudosonvex at x^*, $\hat{\varphi}_j$ is increasing, and $\hat{\varphi}_j(0) = 0$;

(iii) for each $k \in K_* \equiv K_*(w^*)$, $\xi \to \mathcal{D}_k(\xi, w^*)$ is $(\breve{\varphi}_k, \eta, \breve{\rho}_k, \theta, m)$-quasisonvex at x^*, and $\breve{\varphi}_k(0) = 0$;

(iv) $\rho^\circ(x, x^*) + \sum_{j\in J_+} v_j^*\hat{\rho}_j(x, x^*) + \sum_{k\in K_*} \breve{\rho}_k(x, x^*) \geq 0$ for all $x \in \mathbb{F}$, where $\rho^\circ(x, x^*) = \sum_{i\in I_+} u_i^*\bar{\rho}_i(x, x^*)$;

(b) (i) for each $i \in I_+$, $\xi \to \mathcal{E}_i(\xi, x^*, u^*)$ is prestrictly $(\bar{\varphi}_i, \eta, \bar{\rho}_i, \theta, m)$-quasisonvex at x^*, $\bar{\varphi}_i$ is strictly increasing, and $\bar{\varphi}_i(0) = 0$;

(ii) for each $j \in J_+$, G_j is $(\hat{\varphi}_j, \eta, \hat{\rho}_j, \theta, m)$-quasisonvex at x^*, $\hat{\varphi}_j$ is increasing, and $\hat{\varphi}_j(0) = 0$;

(iii) for each $k \in K_*$, $\xi \to \mathcal{D}_k(\xi, w^*)$ is strictly $(\breve{\varphi}_k, \eta, \breve{\rho}_k, \theta, m)$-pseudosonvex at x^*, and $\breve{\varphi}_k(0) = 0$;

(iv) $\rho^\circ(x, x^*) + \sum_{j\in J_+} v_j^* \hat{\rho}_j(x, x^*) + \sum_{k\in K_*} \breve{\rho}_k(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(c) (i) for each $i \in I_+$, $\xi \to \mathcal{E}_i(\xi, x^*, u^*)$ is prestrictly $(\bar{\varphi}_i, \eta, \bar{\rho}_i, \theta, m)$-quasisonvex at x^*, $\bar{\varphi}_i$ is strictly increasing, and $\bar{\varphi}_i(0) = 0$;

(ii) $\xi \to \mathcal{C}(\xi, v^*)$ is strictly $(\hat{\varphi}, \eta, \hat{\rho}, \theta, m)$-pseudosonvex at x^*, $\hat{\varphi}$ is increasing, and $\hat{\varphi}(0) = 0$;

(iii) for each $k \in K_*$, $\xi \to \mathcal{D}_k(\xi, w^*)$ is $(\breve{\varphi}_k, \eta, \breve{\rho}_k, \theta, m)$-quasisonvex at x^*, and $\breve{\varphi}_k(0) = 0$;

(iv) $\rho^\circ(x, x^*) + \hat{\rho}(x, x^*) + \sum_{k\in K_*} \breve{\rho}_k(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(d) (i) for each $i \in I_+$, $\xi \to \mathcal{E}_i(\xi, x^*, u^*)$ is prestrictly $(\bar{\varphi}_i, \eta, \bar{\rho}_i, \theta, m)$-quasisonvex at x^*, $\bar{\varphi}_i$ is strictly increasing, and $\bar{\varphi}_i(0) = 0$;

(ii) for each $j \in J_+$, G_j is $(\hat{\varphi}_j, \eta, \hat{\rho}_j, \theta, m)$-quasisonvex at x^*, $\hat{\varphi}_j$ is increasing, and $\hat{\varphi}_j(0) = 0$;

(iii) $\xi \to \mathcal{D}(\xi, w^*)$ is strictly $(\breve{\varphi}, \eta, \breve{\rho}, \theta, m)$-pseudosonvex at x^*, and $\breve{\varphi}(0) = 0$;

(iv) $\rho^\circ(x, x^*) + \sum_{j\in J_+} v_j^* \hat{\rho}_j(x, x^*) + \breve{\rho}(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(e) (i) for each $i \in I_+$, $\xi \to \mathcal{E}_i(\xi, x^*, u^*)$ is prestrictly $(\bar{\varphi}_i, \eta, \bar{\rho}_i, \theta, m)$-quasisonvex at x^*, $\bar{\varphi}_i$ is strictly increasing, and $\bar{\varphi}_i(0) = 0$;

(ii) $\xi \to \mathcal{C}(\xi, v^*)$ is strictly $(\hat{\varphi}, \eta, \hat{\rho}, \theta, m)$-pseudosonvex at x^*, $\hat{\varphi}$ is increasing, and $\hat{\varphi}(0) = 0$;

(iii) $\xi \to \mathcal{D}(\xi, w^*)$ is $(\breve{\varphi}, \eta, \breve{\rho}, \theta, m)$-quasisonvex at x^*, and $\breve{\varphi}(0) = 0$;

(iv) $\rho^\circ(x, x^*) + \hat{\rho}(x, x^*) + \breve{\rho}(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(f) (i) for each $i \in I_+$, $\xi \to \mathcal{E}_i(\xi, x^*, u^*)$ is prestrictly $(\bar{\varphi}_i, \eta, \bar{\rho}_i, \theta, m)$-quasisonvex at x^*, $\bar{\varphi}_i$ is strictly increasing, and $\bar{\varphi}_i(0) = 0$;

(ii) $\xi \to \mathcal{C}(\xi, v^*)$ is $(\hat{\varphi}, \eta, \hat{\rho}, \theta, m)$-quasisonvex at x^*, $\hat{\varphi}$ is increasing, and $\hat{\varphi}(0) = 0$;

(iii) $\xi \to \mathcal{D}(\xi, w^*)$ is strictly $(\breve{\varphi}, \eta, \breve{\rho}, \theta, m)$-pseudosonvex at x^*, and $\breve{\varphi}(0) = 0$;

(iv) $\rho^\circ(x, x^*) + \hat{\rho}(x, x^*) + \breve{\rho}(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(g) (i) for each $i \in I_+$, $\xi \to \mathcal{E}_i(\xi, x^*, u^*)$ is prestrictly $(\bar{\varphi}_i, \eta, \bar{\rho}_i, \theta, m)$-quasisonvex at x^*, $\bar{\varphi}_i$ is strictly increasing, and $\bar{\varphi}_i(0) = 0$;

(ii) $\xi \to \mathcal{G}(\xi, v^*, w^*)$ is strictly $(\hat{\varphi}, \eta, \hat{\rho}, \theta, m)$-pseudosonvex at x^*, $\hat{\varphi}$ is increasing, and $\hat{\varphi}(0) = 0$;

(iii) $\rho^\circ(x, x^*) + \hat{\rho}(x, x^*) \geq 0$ for all $x \in \mathbb{F}$.

Then x^* is an optimal solution of (P).

Proof. The proof is similar to that of Theorem 2.4. Q.E.D.

Theorem 2.8. Let $x^* \in \mathbb{F}$ and assume that the functions $f_i,\ g_i,\ i \in \underline{p},\ G_j,\ j \in \underline{q}$, and $H_k,\ k \in \underline{r}$, are twice differentiable at x^*, and that for each $z^* \in C(x^*)$, there exist $u^* \in \bar{U},\ v^* \in \mathbb{R}^q_+$, and $w^* \in \mathbb{R}^r$ such that (2.1) - (2.5) hold. Assume, furthermore, that any one of the following five sets of hypotheses is satisfied:

(a) (i) for each $i \in I_+ \equiv I_+(u^*),\ \xi \to \mathcal{E}_i(\xi, x^*, u^*)$ is $(\bar{\varphi}_i, \eta, \bar{\rho}_i, \theta, m)$-pseudosonvex at $x^*,\ \bar{\varphi}_i$ is strictly increasing, and $\bar{\varphi}_i(0) = 0$;

(ii) for each $j \in J_+ \equiv J_+(v^*), G_j$ is $(\hat{\varphi}_j, \eta, \hat{\rho}_j, \theta, m)$-quasisonvex at $x^*,\ \hat{\varphi}_j$ is increasing, and $\hat{\varphi}_j(0) = 0$;

(iii) for each $k \in K_* \equiv K_*(w^*),\ \xi \to \mathcal{D}_k(\xi, w^*)$ is $(\breve{\varphi}_k, \eta, \breve{\rho}_k, \theta, m)$-quasisonvex at x^*, and $\breve{\varphi}_k(0) = 0$;

(iv) $\rho^\circ(x, x^*) + \sum_{j \in J_+} v_j^* \hat{\rho}_j(x, x^*) + \sum_{k \in K_*} \breve{\rho}_k(x, x^*) \geq 0$ for all $x \in \mathbb{F}$, where $\rho^\circ(x, x^*) = \sum_{i \in I_+} u_i^* \bar{\rho}_i(x, x^*)$;

(b) (i) for each $i \in I_+,\ \xi \to \mathcal{E}_i(\xi, x^*, u^*)$ is $(\bar{\varphi}_i, \eta, \bar{\rho}_i, \theta, m)$-pseudosonvex at $x^*,\ \bar{\varphi}_i$ is strictly increasing, and $\bar{\varphi}_i(0) = 0$;

(ii) $\xi \to \mathcal{C}(\xi, v^*)$ is $(\hat{\varphi}, \eta, \hat{\rho}, \theta, m)$-quasisonvex at $x^*,\ \hat{\varphi}$ is increasing, and $\hat{\varphi}(0) = 0$;

(iii) for each $k \in K_*,\ \xi \to \mathcal{D}_k(\xi, w^*)$ is $(\breve{\varphi}_k, \eta, \breve{\rho}_k, \theta, m)$-quasisonvex at x^*, and $\breve{\varphi}_k(0) = 0$;

(iv) $\rho^\circ(x, x^*) + \hat{\rho}(x, x^*) + \sum_{k \in K_*} \breve{\rho}_m(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(c) (i) for each $i \in I_+,\ \xi \to \mathcal{E}_i(\xi, x^*, u^*)$ is $(\bar{\varphi}_i, \eta, \bar{\rho}_i, \theta, m)$-pseudosonvex at $x^*,\ \bar{\varphi}_i$ is strictly increasing, and $\bar{\varphi}_i(0) = 0$;

(ii) for each $j \in J_+,\ G_j$ is $(\hat{\varphi}_j, \eta, \hat{\rho}_j, \theta, m)$-quasisonvex at $x^*,\ \hat{\varphi}_j$ is increasing, and $\hat{\varphi}_j(0) = 0$;

(iii) $\xi \to \mathcal{D}(\xi, w^*)$ is $(\breve{\varphi}, \eta, \breve{\rho}, \theta, m)$-quasisonvex at x^*, and $\breve{\varphi}(0) = 0$;

(iv) $\rho^\circ(x, x^*) + \sum_{j \in J_+} v_j^* \hat{\rho}_j(x, x^*) + \breve{\rho}(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(d) (i) for each $i \in I_+,\ \xi \to \mathcal{E}_i(\xi, x^*, u^*)$ is $(\bar{\varphi}_i, \eta, \bar{\rho}_i, \theta, m)$-pseudosonvex at $x^*,\ \bar{\varphi}_i$ is strictly increasing, and $\bar{\varphi}_i(0) = 0$;

(ii) $\xi \to \mathcal{C}(\xi, v^*)$ is $(\hat{\varphi}, \eta, \hat{\rho}, \theta, m)$-quasisonvex at $x^*,\ \hat{\varphi}$ is increasing, and $\hat{\varphi}(0) = 0$;

(iii) $\xi \to \mathcal{D}(\xi, v^*)$ is $(\breve{\varphi}, \eta, \breve{\rho}, \theta, m)$-quasisonvex at x^*, and $\breve{\varphi}(0) = 0$;

(iv) $\rho^\circ(x, x^*) + \hat{\rho}(x, x^*) + \breve{\rho}(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(e) (i) for each $i \in I_+,\ \xi \to \mathcal{E}_i(\xi, x^*, u^*)$ is $(\bar{\varphi}_i, \eta, \bar{\rho}_i, \theta, m)$-pseudosonvex at $x^*,\ \bar{\varphi}_i$ is strictly increasing, and $\bar{\varphi}_i(0) = 0$;

(ii) $\xi \to \mathcal{G}(\xi, v^*, w^*)$ is $(\hat{\varphi}, \eta, \hat{\rho}, \theta, m)$-quasisonvex at $x^*,\ \hat{\varphi}$ is increasing, and $\hat{\varphi}(0) = 0$;

(iii) $\rho^\circ(x, x^*) + \hat{\rho}(x, x^*) \geq 0$ for all $x \in \mathbb{F}$.

Then x^* is an optimal solution of (P).

Proof. (a) : Suppose that x^* is not an optimal solution of (P). This implies that there exists $\bar{x} \in \mathbb{F}$ such that for each $i \in \underline{p}$, $f_i(\bar{x}) - \lambda^* g_i(\bar{x}) < 0$ and hence $\bar{\varphi}_i\big(\mathcal{E}_i(\bar{x}, x^*, u^*) - \mathcal{E}_i(x^*, x^*, u^*)\big) < 0$ because $\mathcal{E}_i(x^*, x^*, u^*) = 0$ by (2.3). In view of (i), this implies that for each $i \in I_+$, we have

$$\langle D(x^*, u^*)\nabla f_i(x^*) - N(x^*, u^*)\nabla g_i(x^*), \eta(\bar{x}, x^*)\rangle + \frac{1}{2}\langle z^*, [D(x^*, u^*)\nabla^2 f_i(x^*) - N(x^*, u^*)\nabla^2 g_i(x^*)]z^*\rangle < -\bar{\rho}_i(\bar{x}, x^*)\|\theta(\bar{x}, x^*)\|^m.$$

Since $u^* \geq 0$ and $\sum_{i=1}^p u_i^* = 1$, the above inequalities yield

$$\Big\langle \sum_{i=1}^p u_i^*[D(x^*, u^*)\nabla f_i(x^*) - N(x^*, u^*)\nabla g_i(x^*)], \eta(\bar{x}, x^*)\Big\rangle + \frac{1}{2}\Big\langle z^*, \sum_{i=1}^p u_i^*[D(x^*, u^*)\nabla^2 f_i(x^*) - N(x^*, u^*)\nabla^2 g_i(x^*)]z^*\Big\rangle < -\sum_{i\in I_+} u_i^*\bar{\rho}_i(\bar{x}, x^*)\|\theta(\bar{x}, x^*)\|^m. \quad (2.11)$$

Now combining this inequality with (2.7) and (2.8), which are valid for the present case because of the assumptions set forth in (ii) and (iii), and using (iv), we get

$$\Big\langle \sum_{i=1}^p u_i^*[D(x^*, u^*)f_i(x^*) - N(x^*, u^*)\nabla g_i(x^*)], \eta(\bar{x}, x^*)\Big\rangle + \frac{1}{2}\Big\langle z^*, \sum_{i=1}^p u_i^*[D(x^*, u^*)\nabla^2 f_i(x^*) - N(x^*, u^*)\nabla^2 g_i(x^*)]z^*\Big\rangle \geq \Big[\sum_{j\in J_+} v_j^*\hat{\rho}_j(x, x^*) + \sum_{k\in K_*}\breve{\rho}_k(x, x^*)\Big]\|\theta(\bar{x}, x^*)\|^m$$
$$\geq -\sum_{i\in I_+}\bar{\rho}_i(\bar{x}, x^*)\|\theta(\bar{x}, x^*)\|^m,$$

which contradicts (2.11). Therefore, we conclude that x^* is an optimal solution of (P).

(b) - (e) : The proofs are similar to that of part (a). Q.E.D.

In the remainder of this section, we briefly discuss certain modifications of Theorems 2.1 - 2.7 obtained by replacing (2.1) with a certain inequality. We begin by stating the following variant of Theorem 2.1; its proof is almost identical to that of Theorem 2.1 and hence omitted.

Theorem 2.9. Let $x^* \in \mathbb{F}$ and assume that the functions $f_i,\ g_i,\ i \in \underline{p},\ G_j,\ j \in \underline{q}$, and $H_k,\ k \in \underline{r}$, are twice differentiable at x^*, and that for each $z^* \in C(x^*)$, there exist $u^* \in U$, $v^* \in \mathbb{R}^q_+$, and $w^* \in \mathbb{R}^r$ such that (2.2) - (2.5) and the following inequality hold:

$$\Big\langle \sum_{i=1}^p u_i^*[D(x^*, u^*)\nabla f_i(x^*) - N(x^*, u^*)\nabla g_i(x^*)] + \sum_{j=1}^q v_j^*\nabla G_j(x^*) + \sum_{k=1}^r w_k^*\nabla H_k(x^*), \eta(x, x^*)\Big\rangle \geq 0 \ \text{ for all } x \in \mathbb{F}, \quad (2.12)$$

where $\eta : X \times X \to \mathbb{R}^n$ is a given function. Furthermore, assume that any one of the six sets of conditions specified in Theorem 2.1 is satisfied. Then x^* is an optimal solution of (P).

Although the proofs of Theorems 2.1 and 2.8 are essentially the same, their contents are somewhat different. This can easily be seen by comparing (2.1) with (2.12). We observe that any triple (x^*, z^*, λ^*) that satisfies (2.1) - (2.5) also satisfies (2.2) - (2.5) and (2.12), but the converse is not necessarily true. Moreover, (2.1) is a system of n equations, whereas (2.12) is a single inequality. Evidently, from a computational point of view, (2.1) is preferable to (2.12) because of the dependence of the latter on the feasible set of (P).

The modified versions of Theorems 2.2 - 2.7 can be stated in a similar manner.

3 Concluding remarks

Based on a direct nonparametric approach, in this paper we have established numerous sets of parameter-free second-order sufficient optimality criteria for a discrete minmax fractional programming problem using a variety of generalized $(\varphi, \eta, \rho, \theta, m)$-sonvexity assumptions. These optimality results can be used for constructing various duality models as well as for developing new algorithms for the numerical solution of discrete minmax fractional programming problems. Furthermore, the obtained results in this paper can be applied in studying other related classes of nonlinear programming problems, especially to the second-order sufficient optimality aspects of the following 'semiinfinite' minmax fractional programming problem:

$$\text{Minimize } \max_{1\leq i\leq p} \frac{f_i(x)}{g_i(x)}$$

subject to

$$G_j(x,t) \leq 0 \text{ for all } t \in T_j, \quad j \in \underline{q},$$
$$H_k(x,s) = 0 \text{ for all } s \in S_k, \quad k \in \underline{r},$$
$$x \in X,$$

where X, f_i, and g_i, $i \in \underline{p}$, are as defined in the description of (P), for each $j \in \underline{q}$ and $k \in \underline{r}$, T_j and S_k are compact subsets of complete metric spaces, for each $j \in \underline{q}$, $\xi \to G_j(\xi, t)$ is a real-valued function defined on X for all $t \in T_j$, for each $k \in \underline{r}$, $\xi \to H_k(\xi, s)$ is a real-valued function defined on X for all $s \in S_k$, for each $j \in \underline{q}$ and $k \in \underline{r}$, $t \to G_j(x,t)$ and $s \to H_k(x,s)$ are continuous real-valued functions defined, respectively, on T_j and S_k for all $x \in X$.

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